Hence,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Suppose PQ is a diameter and $\angle SPQ = \alpha$ and $\angle RPQ = \beta$. (Fig. 2.) Then

$$PR \cdot QS = PS \cdot RQ + PQ \cdot SR$$

or

$$2r\cos\beta \cdot 2r\sin\alpha = 2r\cos\alpha \cdot 2r\sin\beta + 2r \cdot 2r\sin(\alpha-\beta).$$

Hence,

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

Suppose PQ is a diameter and $\angle RPQ = \alpha$ and $\angle SQP = \beta$. (Fig. 3.)

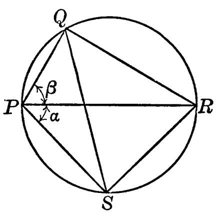

FIG. 1.

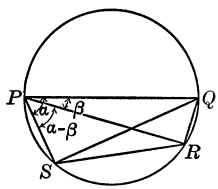

FIG. 2.

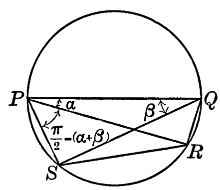

FIG. 3.

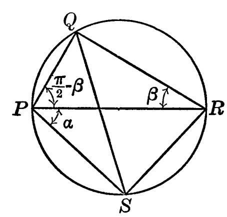

FIG. 4.

Then

$$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$$

or

$$2r\cos\alpha \cdot 2r\cos\beta = 2r\sin\beta \cdot 2r\sin\alpha + 2r \cdot 2r\sin[\pi/2 - (\alpha+\beta)].$$

Hence,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Suppose PR is a diameter and $\angle SPR = \alpha$ and $\angle PRQ = \beta$. (Fig. 4.) Then

$$PR \cdot SQ = PS \cdot RQ + PQ \cdot SR$$

or

$$2r \cdot 2r\sin[\pi/2 + (\alpha-\beta)] = 2r\cos\alpha \cdot 2r\cos\beta + 2r\sin\beta \cdot 2r\sin\alpha.$$

Hence,

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Also solved by C. N. SCHMALL.

439. Proposed by CHARLES N. SCHMALL, New York City.

Show that the areas of any two triangles circumscribed about the same circle are in the same ratio as their perimeters.

SOLUTION BY A. L. MCCARTY, Cape Girardeau, Mo.

Let the radius of the circle be r and the sides of the circumscribed triangles be a, b, c and d, e, f respectively.

Now it is evident that the area of the first triangle is $\frac{1}{2}r(a+b+c)$ and the area of the second is $\frac{1}{2}r(d+e+f)$. Hence, the two triangles are to each other as their perimeters.

Also solved by HORACE OLSON, J. L. RILEY, H. C. FEEMSTER, C. E. FLANAGAN, CLIFFORD N. MILLS, C. E. GITHENS, GEO. W. HARTWELL, ELMER SCHUYLER, WALTER C. EELLS, and A. M. HARDING.

MECHANICS.

288. Proposed by C. E. HORNE, Westminister College, Colorado.

Show that the tangential velocity of a projectile at any point of its path is equal to the velocity it would have acquired in falling, under the influence of gravitation alonè, from the directrix to the point in question.

SOLUTION BY A. M. HARDING, University of Arkansas.

The equation of the path in parametric form is

$$x = u \cos \alpha \cdot t, \qquad y = u \sin \alpha \cdot t - \tfrac{1}{2}gt^2,$$

where u is the initial velocity and α is the angle of projection.

Then

$$dx = u \cos \alpha \, dt, \qquad dy = (u \sin \alpha - gt)dt,$$

$$ds^2 = dx^2 + dy^2 = (u^2 - 2ug \sin \alpha \cdot t + g^2t^2)dt^2 = (u^2 - 2gy)dt^2.$$

Now it can be easily shown that the distance from the directrix to the X-axis is given by $d = u^2/2g$. Hence $ds^2 = (2gd - 2gy)dt^2$.

Whence

$$\frac{ds}{dt} = \text{velocity} = \sqrt{2g(d - y)}.$$

Hence, the velocity at any point is equal to the velocity it would have acquired in falling from the directrix.

Also solved by ELIJAH SWIFT, CLIFFORD N. MILLS, HORACE OLSON, J. W. CLAWSON.

NUMBER THEORY.

207. Proposed by A. J. KEMPNER, University of Illinois.

There are 80 positive integers < 100 containing no figure 9 against 19 containing at least one figure 9. (For integers < 1000 the numbers are 728 and 271 respectively.) One might be led to believe that for every positive integer M the number N_1 of positive integers $< M$ containing no figure 9 is always greater than the number N_2 of positive integers $< M$ containing at least one figure 9.

To prove: $\lim (N_1/N_2) = 0$ for $M \doteq \infty$.

SOLUTION BY LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Let N_2 in every case represent the number of positive integers from 1 to M inclusive, which contain the figure 9 at least once; while N_1 represents the number of positive integers from 1 to M inclusive, which do not contain the figure 9. Then for

$M = 10, \qquad N_2 = 1,$

$M = 10^2, \qquad N_2 = 9 \times 1 + 10, \text{ or } 9 + 10.$

$M = 10^3, \qquad N_2 = [9 \times 1 + 10]9 + 10^2, \quad \text{or } 9^2 + 9 \cdot 10 + 10^2.$